4.3 Green's Theorem

We will now see a way of evaluating the line integral of a *smooth* vector field around a simple closed curve. A vector field $\mathbf{f}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ is **smooth** if its component functions $P(x,y)$ and $Q(x,y)$ are smooth. We will use *Green's Theorem* (sometimes called *Green's Theorem in the plane*) to relate the *line* integral around a closed curve with a *double* integral over the region inside the curve:

Theorem 4.7. **(Green's Theorem)** Let R be a region in $\mathbb{R}^2$ whose boundary is a simple closed curve C which is piecewise smooth. Let $\mathbf{f}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ be a smooth vector field defined on both R and C. Then

$$\oint_C \mathbf{f} \cdot d\mathbf{r} \;=\; \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA\,, \tag{4.21}$$

where C is traversed so that R is always on the left side of C.

Proof: We will prove the theorem in the case for a *simple* region R, that is, where the boundary curve C can be written as $C = C_1 \cup C_2$ in two distinct ways:

$$C_1 = \text{the curve } y = y_1(x) \text{ from the point } X_1 \text{ to the point } X_2 \tag{4.22}$$

$$C_2 = \text{the curve } y = y_2(x) \text{ from the point } X_2 \text{ to the point } X_1, \tag{4.23}$$

where X_1 and X_2 are the points on C farthest to the left and right, respectively; and

$$C_1 = \text{the curve } x = x_1(y) \text{ from the point } Y_2 \text{ to the point } Y_1 \tag{4.24}$$

$$C_2 = \text{the curve } x = x_2(y) \text{ from the point } Y_1 \text{ to the point } Y_2, \tag{4.25}$$

where Y_1 and Y_2 are the lowest and highest points, respectively, on C. See Figure 4.3.1.

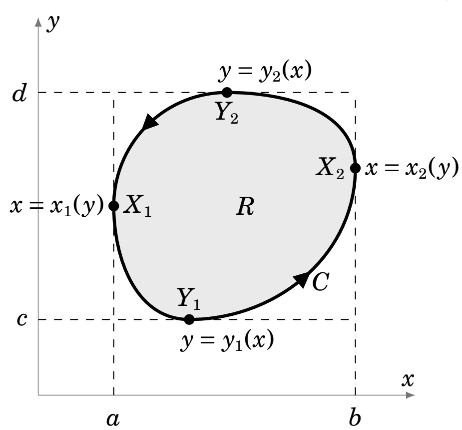


Figure 4.3.1

Integrate $P(x,y)$ around C using the representation $C = C_1 \cup C_2$ given by (4.23) and (4.24).

Since $y = y_1(x)$ along C_1 (as x goes from a to b) and $y = y_2(x)$ along C_2 (as x goes from b to a), as we see from Figure 4.3.1, then we have

$$\begin{aligned}
\oint_C P(x,y)\,dx &= \int_{C_1} P(x,y)\,dx + \int_{C_2} P(x,y)\,dx \\
&= \int_a^b P(x,y_1(x))\,dx + \int_b^a P(x,y_2(x))\,dx \\
&= \int_a^b P(x,y_1(x))\,dx - \int_a^b P(x,y_2(x))\,dx \\
&= -\int_a^b (P(x,y_2(x)) - P(x,y_1(x)))\,dx \\
&= -\int_a^b \left(P(x,y)\Big|_{y=y_1(x)}^{y=y_2(x)} \right) dx \\
&= -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial P(x,y)}{\partial y}\,dy\,dx \quad \text{(by the Fundamental Theorem of Calculus)} \\
&= -\iint_R \frac{\partial P}{\partial y}\,dA\,.
\end{aligned}$$

Likewise, integrate $Q(x,y)$ around C using the representation $C = C_1 \cup C_2$ given by (4.25) and (4.26). Since $x = x_1(y)$ along C_1 (as y goes from d to c) and $x = x_2(y)$ along C_2 (as y goes from c to d), as we see from Figure 4.3.1, then we have

$$\begin{aligned}
\oint_C Q(x,y)\,dy &= \int_{C_1} Q(x,y)\,dy + \int_{C_2} Q(x,y)\,dy \\
&= \int_d^c Q(x_1(y),y)\,dy + \int_c^d Q(x_2(y),y)\,dy \\
&= -\int_c^d Q(x_1(y),y)\,dy + \int_c^d Q(x_2(y),y)\,dy \\
&= \int_c^d (Q(x_2(y),y) - Q(x_1(y),y))\,dy \\
&= \int_c^d \left(Q(x,y)\Big|_{x=x_1(y)}^{x=x_2(y)} \right) dy \\
&= \int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial Q(x,y)}{\partial x}\,dx\,dy \quad \text{(by the Fundamental Theorem of Calculus)} \\
&= \iint_R \frac{\partial Q}{\partial x}\,dA\,, \text{ and so}
\end{aligned}$$

$$\begin{aligned}\oint_C \mathbf{f}\cdot d\mathbf{r} &= \oint_C P(x,y)\,dx + \oint_C Q(x,y)\,dy\\ &= -\iint\limits_R \frac{\partial P}{\partial y}\,dA + \iint\limits_R \frac{\partial Q}{\partial x}\,dA\\ &= \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA\ .\end{aligned}$$

QED

Though we proved Green's Theorem only for a simple region R, the theorem can also be proved for more general regions (say, a union of simple regions).[2]

Example 4.7. Evaluate $\oint_C (x^2+y^2)\,dx + 2xy\,dy$, where C is the boundary (traversed counterclockwise) of the region $R = \{(x,y) : 0 \le x \le 1,\ 2x^2 \le y \le 2x\}$.

Solution: R is the shaded region in Figure 4.3.2. By Green's Theorem, for $P(x,y) = x^2+y^2$ and $Q(x,y) = 2xy$, we have

$$\begin{aligned}\oint_C (x^2+y^2)\,dx + 2xy\,dy &= \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA\\ &= \iint\limits_R (2y-2y)\,dA \;=\; \iint\limits_R 0\,dA \;=\; 0\ .\end{aligned}$$

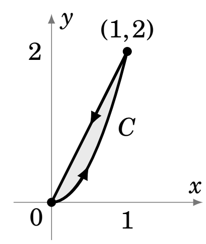


Figure 4.3.2

We actually already knew that the answer was zero. Recall from Example 4.5 in Section 4.2 that the vector field $\mathbf{f}(x,y) = (x^2+y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ has a potential function $F(x,y) = \frac{1}{3}x^3 + xy^2$, and so $\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$ by Corollary 4.6.

Example 4.8. Let $\mathbf{f}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$, where

$$P(x,y) = \frac{-y}{x^2+y^2} \quad\text{and}\quad Q(x,y) = \frac{x}{x^2+y^2}\ ,$$

and let $R = \{(x,y) : 0 < x^2+y^2 \le 1\}$. For the boundary curve $C : x^2+y^2 = 1$, traversed counterclockwise, it was shown in Exercise 9(b) in Section 4.2 that $\oint_C \mathbf{f}\cdot d\mathbf{r} = 2\pi$. But

$$\frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial P}{\partial y} \;\Rightarrow\; \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint\limits_R 0\,dA = 0\ .$$

This would seem to contradict Green's Theorem. However, note that R is not the *entire* region enclosed by C, since the point $(0,0)$ is not contained in R. That is, R has a "hole" at the origin, so Green's Theorem does not apply.

[2]See TAYLOR and MANN, § 15.31 for a discussion of some of the difficulties involved when the boundary curve is "complicated".

If we modify the region R to be the *annulus* $R = \{(x,y) : 1/4 \le x^2 + y^2 \le 1\}$ (see Figure 4.3.3), and take the "boundary" C of R to be $C = C_1 \cup C_2$, where C_1 is the unit circle $x^2 + y^2 = 1$ traversed counterclockwise and C_2 is the circle $x^2 + y^2 = 1/4$ traversed *clockwise*, then it can be shown (see Exercise 8) that

$$\oint_C \mathbf{f} \cdot d\mathbf{r} \;=\; 0\,.$$

We would still have $\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0$, so for this R we would have

$$\oint_C \mathbf{f} \cdot d\mathbf{r} \;=\; \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA\,,$$

which shows that Green's Theorem holds for the annular region R.

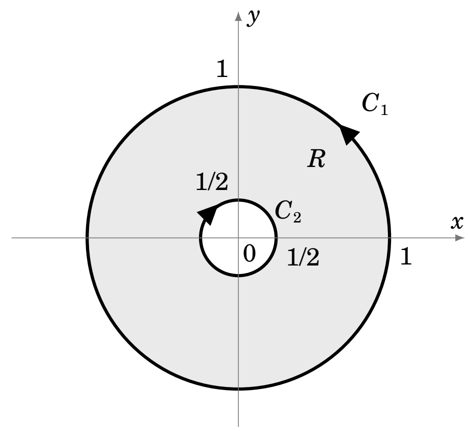
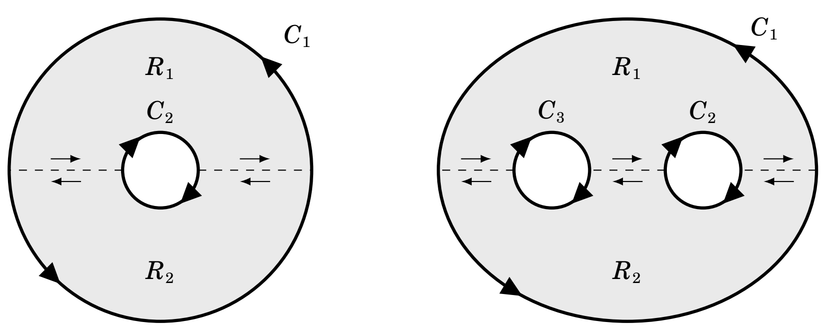


Figure 4.3.3 The annulus R

It turns out that Green's Theorem can be extended to *multiply connected* regions, that is, regions like the annulus in Example 4.8, which have one or more *regions* cut out from the interior, as opposed to discrete *points* being cut out. For such regions, the "outer" boundary and the "inner" boundaries are traversed so that R is always on the left side.



(a) Region R with one hole **(b)** Region R with two holes

Figure 4.3.4 Multiply connected regions

The intuitive idea for why Green's Theorem holds for multiply connected regions is shown in Figure 4.3.4 above. The idea is to cut "slits" between the boundaries of a multiply connected region R so that R is divided into subregions which do *not* have any "holes". For example, in Figure 4.3.4(a) the region R is the union of the regions R_1 and R_2, which are divided by the slits indicated by the dashed lines. Those slits are part of the boundary of both R_1 and R_2, and we traverse then in the manner indicated by the arrows. Notice that along each slit the boundary of R_1 is traversed in the opposite direction as that of R_2, which

means that the line integrals of $\mathbf{f}$ along those slits cancel each other out. Since R_1 and R_2 do not have holes in them, then Green's Theorem holds in each subregion, so that

$$\oint_{\substack{\text{bdy}\\ \text{of } R_1}} \mathbf{f} \cdot d\mathbf{r} = \iint\limits_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad \text{and} \quad \oint_{\substack{\text{bdy}\\ \text{of } R_2}} \mathbf{f} \cdot d\mathbf{r} = \iint\limits_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \; .$$

But since the line integrals along the slits cancel out, we have

$$\oint_{C_1 \cup C_2} \mathbf{f} \cdot d\mathbf{r} = \oint_{\substack{\text{bdy}\\ \text{of } R_1}} \mathbf{f} \cdot d\mathbf{r} + \oint_{\substack{\text{bdy}\\ \text{of } R_2}} \mathbf{f} \cdot d\mathbf{r} \, ,$$

and so

$$\oint_{C_1 \cup C_2} \mathbf{f} \cdot d\mathbf{r} = \iint\limits_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint\limits_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint\limits_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \, ,$$

which shows that Green's Theorem holds in the region R. A similar argument shows that the theorem holds in the region with two holes shown in Figure 4.3.4(b).

We know from Corollary 4.6 that when a smooth vector field $\mathbf{f}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ on a region R (whose boundary is a piecewise smooth, simple closed curve C) has a potential in R, then $\oint_C \mathbf{f} \cdot d\mathbf{r} = 0$. And if the potential $F(x, y)$ is smooth in R, then $\frac{\partial F}{\partial x} = P$ and $\frac{\partial F}{\partial y} = Q$, and so we know that

$$\frac{\partial^2 F}{\partial y \, \partial x} = \frac{\partial^2 F}{\partial x \, \partial y} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{in } R.$$

Conversely, if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in R then

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint\limits_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint\limits_{R} 0 \, dA = 0 \, .$$

For a **simply connected** region R (i.e. a region with no holes), the following can be shown:

The following statements are equivalent for a simply connected region R in $\mathbb{R}^2$:

(a) $\mathbf{f}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ has a smooth potential $F(x, y)$ in R

(b) $\displaystyle\int_C \mathbf{f} \cdot d\mathbf{r}$ is independent of the path for any curve C in R

(c) $\displaystyle\oint_C \mathbf{f} \cdot d\mathbf{r} = 0$ for every simple closed curve C in R

(d) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ in R (in this case, the differential form $P\,dx + Q\,dy$ is exact)

Exercises

A

For Exercises 1-4, use Green's Theorem to evaluate the given line integral around the curve C, traversed counterclockwise.

1. $\oint_C (x^2 - y^2)\,dx + 2xy\,dy$; C is the boundary of $R = \{(x,y) : 0 \le x \le 1,\ 2x^2 \le y \le 2x\}$

2. $\oint_C x^2 y\,dx + 2xy\,dy$; C is the boundary of $R = \{(x,y) : 0 \le x \le 1,\ x^2 \le y \le x\}$

3. $\oint_C 2y\,dx - 3x\,dy$; C is the circle $x^2 + y^2 = 1$

4. $\oint_C (e^{x^2} + y^2)\,dx + (e^{y^2} + x^2)\,dy$; C is the boundary of the triangle with vertices $(0,0)$, $(4,0)$ and $(0,4)$

5. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = (y^2 + 3x^2)\,\mathbf{i} + 2xy\,\mathbf{j}$? If so, find one.

6. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = (x^3\cos(xy) + 2x\sin(xy))\,\mathbf{i} + x^2 y\cos(xy)\,\mathbf{j}$? If so, find one.

7. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = (8xy + 3)\,\mathbf{i} + 4(x^2 + y)\,\mathbf{j}$? If so, find one.

8. Show that for any constants a, b and any closed simple curve C, $\oint_C a\,dx + b\,dy = 0$.

B

9. For the vector field $\mathbf{f}$ as in Example 4.8, show directly that $\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$, where C is the boundary of the annulus $R = \{(x,y) : 1/4 \le x^2 + y^2 \le 1\}$ traversed so that R is always on the left.

10. Evaluate $\oint_C e^x \sin y\,dx + (y^3 + e^x \cos y)\,dy$, where C is the boundary of the rectangle with vertices $(1,-1)$, $(1,1)$, $(-1,1)$ and $(-1,-1)$, traversed counterclockwise.

C

11. For a region R bounded by a simple closed curve C, show that the area A of R is

$$A = -\oint_C y\,dx = \oint_C x\,dy = \frac{1}{2}\oint_C x\,dy - y\,dx\,,$$

where C is traversed so that R is always on the left. (*Hint: Use Green's Theorem and the fact that* $A = \iint_R 1\,dA$.)

4.4 Surface Integrals and the Divergence Theorem

In Section 4.1 we learned how to integrate along a curve. We will now learn how to perform integration over a *surface* in $\mathbb{R}^3$, such as a sphere or a paraboloid. Recall from Section 1.8 how we identified points (x,y,z) on a curve C in $\mathbb{R}^3$, parametrized by $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \le t \le b$, with the terminal points of the position vector

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{for } t \text{ in } [a,b].$$

The idea behind a parametrization of a curve is that it "transforms" a subset of $\mathbb{R}^1$ (normally an interval $[a,b]$) into a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ (see Figure 4.4.1).

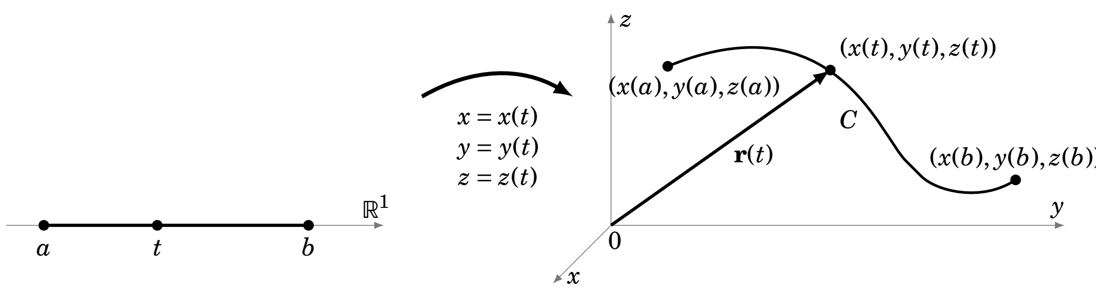


Figure 4.4.1 Parametrization of a curve C in $\mathbb{R}^3$

Similar to how we used a parametrization of a curve to define the line integral along the curve, we will use a parametrization of a surface to define a *surface integral*. We will use *two* variables, u and v, to parametrize a surface Σ in $\mathbb{R}^3$: $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, for (u,v) in some region R in $\mathbb{R}^2$ (see Figure 4.4.2).

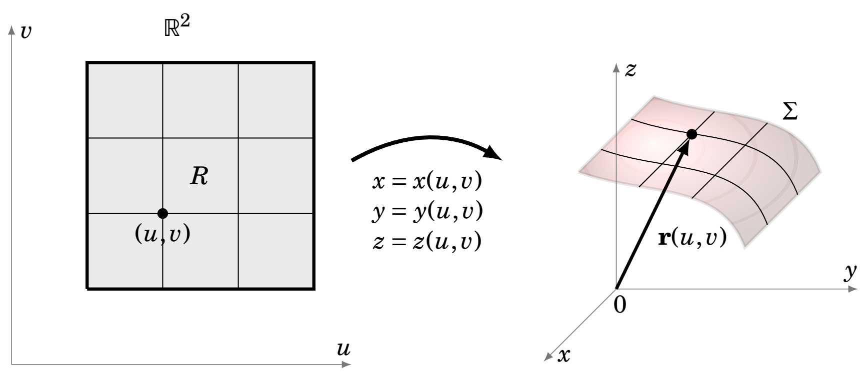


Figure 4.4.2 Parametrization of a surface Σ in $\mathbb{R}^3$

In this case, the position vector of a point on the surface Σ is given by the vector-valued function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \quad \text{for } (u,v) \text{ in } R.$$

Since $\mathbf{r}(u,v)$ is a function of two variables, define the partial derivatives $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ for (u,v) in R by

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u}(u,v) &= \frac{\partial x}{\partial u}(u,v)\mathbf{i} + \frac{\partial y}{\partial u}(u,v)\mathbf{j} + \frac{\partial z}{\partial u}(u,v)\mathbf{k}\,, \text{ and}\\ \frac{\partial \mathbf{r}}{\partial v}(u,v) &= \frac{\partial x}{\partial v}(u,v)\mathbf{i} + \frac{\partial y}{\partial v}(u,v)\mathbf{j} + \frac{\partial z}{\partial v}(u,v)\mathbf{k}\,.\end{aligned}$$

The parametrization of Σ can be thought of as "transforming" a region in $\mathbb{R}^2$ (in the uv-plane) into a 2-dimensional surface in $\mathbb{R}^3$. This parametrization of the surface is sometimes called a *patch*, based on the idea of "patching" the region R onto Σ in the grid-like manner shown in Figure 4.4.2.

In fact, those gridlines in R lead us to how we will define a surface integral over Σ. Along the vertical gridlines in R, the variable u is constant. So those lines get mapped to curves on Σ, and the variable u is constant along the position vector $\mathbf{r}(u,v)$. Thus, the tangent vector to those curves at a point (u,v) is $\frac{\partial \mathbf{r}}{\partial v}$. Similarly, the horizontal gridlines in R get mapped to curves on Σ whose tangent vectors are $\frac{\partial \mathbf{r}}{\partial u}$.

Now take a point (u,v) in R as, say, the lower left corner of one of the rectangular grid sections in R, as shown in Figure 4.4.2. Suppose that this rectangle has a small width and height of Δu and Δv, respectively. The corner points of that rectangle are (u,v), $(u+\Delta u,v)$, $(u+\Delta u,v+\Delta v)$ and $(u,v+\Delta v)$. So the area of that rectangle is $A = \Delta u\,\Delta v$. Then that rectangle gets mapped by the parametrization onto some section of the surface Σ which, for Δu and Δv small enough, will have a surface area (call it $d\sigma$) that is very close to the area of the parallelogram which has adjacent sides $\mathbf{r}(u+\Delta u,v)-\mathbf{r}(u,v)$ (corresponding to the line segment from (u,v) to $(u+\Delta u,v)$ in R) and $\mathbf{r}(u,v+\Delta v)-\mathbf{r}(u,v)$ (corresponding to the line segment from (u,v) to $(u,v+\Delta v)$ in R). But by combining our usual notion of a partial derivative (see Definition 2.3 in Section 2.2) with that of the derivative of a vector-valued function (see Definition 1.12 in Section 1.8) applied to a function of two variables, we have

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} &\approx \frac{\mathbf{r}(u+\Delta u,v)-\mathbf{r}(u,v)}{\Delta u}\,, \text{ and}\\ \frac{\partial \mathbf{r}}{\partial v} &\approx \frac{\mathbf{r}(u,v+\Delta v)-\mathbf{r}(u,v)}{\Delta v}\,,\end{aligned}$$

and so the surface area element $d\sigma$ is approximately

$$\left\|(\mathbf{r}(u+\Delta u,v)-\mathbf{r}(u,v))\times(\mathbf{r}(u,v+\Delta v)-\mathbf{r}(u,v))\right\| \approx \left\|(\Delta u\frac{\partial \mathbf{r}}{\partial u})\times(\Delta v\frac{\partial \mathbf{r}}{\partial v})\right\| = \left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\|\Delta u\,\Delta v$$

by Theorem 1.13 in Section 1.4. Thus, the total surface area S of Σ is approximately the sum of all the quantities $\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\|\Delta u\,\Delta v$, summed over the rectangles in R. Taking the limit of that sum as the diagonal of the largest rectangle goes to 0 gives

$$S = \iint_R \left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv\,. \tag{4.26}$$

We will write the double integral on the right using the special notation

$$\iint\limits_{\Sigma} d\sigma \;=\; \iint\limits_{R} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du\,dv\;. \tag{4.27}$$

This is a special case of a *surface integral* over the surface Σ, where the surface area element $d\sigma$ can be thought of as $1\,d\sigma$. Replacing 1 by a general real-valued function $f(x,y,z)$ defined in $\mathbb{R}^3$, we have the following:

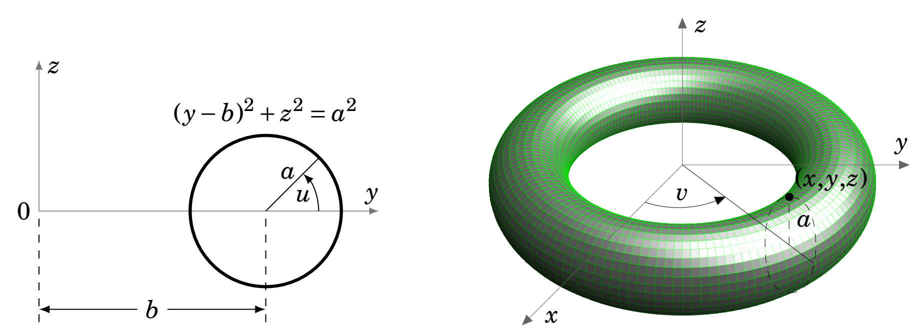
Definition 4.3. Let Σ be a surface in $\mathbb{R}^3$ parametrized by $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, for (u,v) in some region R in $\mathbb{R}^2$. Let $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ be the position vector for any point on Σ, and let $f(x,y,z)$ be a real-valued function defined on some subset of $\mathbb{R}^3$ that contains Σ. The **surface integral** of $f(x,y,z)$ over Σ is

$$\iint\limits_{\Sigma} f(x,y,z)\,d\sigma \;=\; \iint\limits_{R} f(x(u,v),y(u,v),z(u,v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du\,dv\;. \tag{4.28}$$

In particular, the surface area S of Σ is

$$S \;=\; \iint\limits_{\Sigma} 1\,d\sigma\;. \tag{4.29}$$

Example 4.9. A *torus* T is a surface obtained by revolving a circle of radius a in the yz-plane around the z-axis, where the circle's center is at a distance b from the z-axis $(0 < a < b)$, as in Figure 4.4.3. Find the surface area of T.



(a) Circle in the yz-plane

(b) Torus T

Figure 4.4.3

Solution: For any point on the circle, the line segment from the center of the circle to that point makes an angle u with the y-axis in the positive y direction (see Figure 4.4.3(a)). And as the circle revolves around the z-axis, the line segment from the origin to the center of that

circle sweeps out an angle v with the positive x-axis (see Figure 4.4.3(b)). Thus, the torus can be parametrized as:

$$x = (b + a\cos u)\cos v\ , \quad y = (b + a\cos u)\sin v\ , \quad z = a\sin u\ , \quad 0 \le u \le 2\pi\ , \quad 0 \le v \le 2\pi$$

So for the position vector

$$\begin{aligned}\mathbf{r}(u,v) &= x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \\ &= (b + a\cos u)\cos v\,\mathbf{i} + (b + a\cos u)\sin v\,\mathbf{j} + a\sin u\,\mathbf{k}\end{aligned}$$

we see that

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} &= -a\sin u\,\cos v\,\mathbf{i} - a\sin u\,\sin v\,\mathbf{j} + a\cos u\,\mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial v} &= -(b + a\cos u)\sin v\,\mathbf{i} + (b + a\cos u)\cos v\,\mathbf{j} + 0\mathbf{k}\ ,\end{aligned}$$

and so computing the cross product gives

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = -a(b + a\cos u)\cos v\,\cos u\,\mathbf{i} - a(b + a\cos u)\sin v\,\cos u\,\mathbf{j} - a(b + a\cos u)\sin u\,\mathbf{k}\ ,$$

which has magnitude

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = a(b + a\cos u)\ .$$

Thus, the surface area of T is

$$\begin{aligned}S &= \iint_{\Sigma} 1\,d\sigma \\ &= \int_0^{2\pi}\int_0^{2\pi} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du\,dv \\ &= \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos u)\,du\,dv \\ &= \int_0^{2\pi} \left(abu + a^2\sin u \Big|_{u=0}^{u=2\pi} \right) dv \\ &= \int_0^{2\pi} 2\pi ab\,dv \\ &= 4\pi^2 ab\end{aligned}$$

Since $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are tangent to the surface Σ (i.e. lie in the tangent plane to Σ at each point on Σ), then their cross product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is perpendicular to the tangent plane to the surface at each point of Σ. Thus,

$$\iint\limits_{\Sigma} f(x,y,z)\,d\sigma = \iint\limits_{R} f(x(u,v),y(u,v),z(u,v))\,\|\mathbf{n}\|\,d\sigma\ ,$$

where $\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$. We say that $\mathbf{n}$ is a **normal vector** to Σ.

Recall that normal vectors to a plane can point in two opposite directions. By an **outward unit normal vector** to a surface Σ, we will mean the unit vector that is normal to Σ and points away from the "top" (or "outer" part) of the surface. This is a hazy definition, but the picture in Figure 4.4.4 gives a better idea of what outward normal vectors look like, in the case of a sphere. With this idea in mind, we make the following definition of a surface integral of a 3-dimensional *vector field* over a surface:

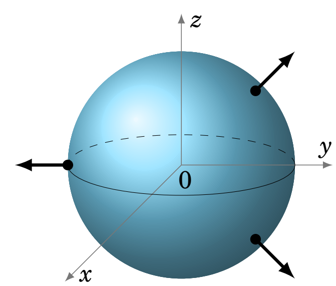


Figure 4.4.4

Definition 4.4. Let Σ be a surface in $\mathbb{R}^3$ and let $\mathbf{f}(x,y,z) = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$ be a vector field defined on some subset of $\mathbb{R}^3$ that contains Σ. The **surface integral** of $\mathbf{f}$ over Σ is

$$\iint\limits_{\Sigma} \mathbf{f}\cdot d\boldsymbol{\sigma} \ = \ \iint\limits_{\Sigma} \mathbf{f}\cdot\mathbf{n}\,d\sigma\ , \tag{4.30}$$

where, at any point on Σ, $\mathbf{n}$ is the outward unit normal vector to Σ.

Note in the above definition that the dot product inside the integral on the right is a real-valued function, and hence we can use Definition 4.3 to evaluate the integral.

Example 4.10. Evaluate the surface integral $\iint\limits_{\Sigma} \mathbf{f}\cdot d\boldsymbol{\sigma}$, where $\mathbf{f}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and Σ is the part of the plane $x+y+z=1$ with $x \ge 0$, $y \ge 0$, and $z \ge 0$, with the outward unit normal $\mathbf{n}$ pointing in the positive z direction (see Figure 4.4.5).

Solution: Since the vector $\mathbf{v} = (1,1,1)$ is normal to the plane $x+y+z=1$ (why?), then dividing $\mathbf{v}$ by its length yields the outward unit normal vector $\mathbf{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. We now need to parametrize Σ. As we can see from Figure 4.4.5, projecting Σ onto the xy-plane yields a triangular region $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$. Thus, using (u,v) instead of (x,y), we see that

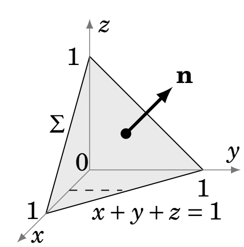


Figure 4.4.5

$$x = u,\ y = v,\ z = 1-(u+v),\quad \text{for } 0 \le u \le 1, 0 \le v \le 1-u$$

is a parametrization of Σ over R (since $z = 1-(x+y)$ on Σ). So on Σ,

$$
\begin{aligned}
\mathbf{f}\cdot\mathbf{n} &= (yz, xz, xy)\cdot\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}(yz+xz+xy) \\
&= \frac{1}{\sqrt{3}}((x+y)z+xy) = \frac{1}{\sqrt{3}}((u+v)(1-(u+v))+uv) \\
&= \frac{1}{\sqrt{3}}((u+v)-(u+v)^2+uv)
\end{aligned}
$$

for (u,v) in R, and for $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} = u\mathbf{i} + v\mathbf{j} + (1-(u+v))\mathbf{k}$ we have

$$
\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (1,0,-1)\times(0,1,-1) = (1,1,1) \quad \Rightarrow \quad \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| = \sqrt{3}\,.
$$

Thus, integrating over R using vertical slices (e.g. as indicated by the dashed line in Figure 4.4.5) gives

$$
\begin{aligned}
\iint_{\Sigma} \mathbf{f}\cdot d\boldsymbol{\sigma} &= \iint_{\Sigma} \mathbf{f}\cdot\mathbf{n}\, d\sigma \\
&= \iint_{R} (\mathbf{f}(x(u,v), y(u,v), z(u,v))\cdot\mathbf{n}) \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| dv\, du \\
&= \int_0^1 \int_0^{1-u} \frac{1}{\sqrt{3}}((u+v)-(u+v)^2+uv)\sqrt{3}\, dv\, du \\
&= \int_0^1 \left(\frac{(u+v)^2}{2} - \frac{(u+v)^3}{3} + \frac{uv^2}{2} \Bigg|_{v=0}^{v=1-u} \right) du \\
&= \int_0^1 \left(\frac{1}{6} + \frac{u}{2} - \frac{3u^2}{2} + \frac{5u^3}{6} \right) du \\
&= \frac{u}{6} + \frac{u^2}{4} - \frac{u^3}{2} + \frac{5u^4}{24} \Bigg|_0^1 = \frac{1}{8}\,.
\end{aligned}
$$

Computing surface integrals can often be tedious, especially when the formula for the outward unit normal vector at each point of Σ changes. The following theorem provides an easier way in the case when Σ is a **closed surface**, that is, when Σ encloses a bounded solid in $\mathbb{R}^3$. For example, spheres, cubes, and ellipsoids are closed surfaces, but planes and paraboloids are not.

Theorem 4.8. (**Divergence Theorem**) Let Σ be a closed surface in $\mathbb{R}^3$ which bounds a solid S, and let $\mathbf{f}(x,y,z) = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$ be a vector field defined on some subset of $\mathbb{R}^3$ that contains Σ. Then

$$\iint\limits_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma} = \iiint\limits_{S} \operatorname{div} \mathbf{f}\, dV , \tag{4.31}$$

where

$$\operatorname{div} \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \tag{4.32}$$

is called the **divergence** of $\mathbf{f}$.

The proof of the Divergence Theorem is very similar to the proof of Green's Theorem, i.e. it is first proved for the simple case when the solid S is bounded above by one surface, bounded below by another surface, and bounded laterally by one or more surfaces. The proof can then be extended to more general solids.[3]

Example 4.11. Evaluate $\iint\limits_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma}$, where $\mathbf{f}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and Σ is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution: We see that $\operatorname{div} \mathbf{f} = 1+1+1 = 3$, so

$$\begin{aligned} \iint\limits_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma} &= \iiint\limits_{S} \operatorname{div} \mathbf{f}\, dV = \iiint\limits_{S} 3\, dV \\ &= 3 \iiint\limits_{S} 1\, dV = 3\operatorname{vol}(S) = 3 \cdot \frac{4\pi(1)^3}{3} = 4\pi . \end{aligned}$$

In physical applications, the surface integral $\iint\limits_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma}$ is often referred to as the **flux** of $\mathbf{f}$ through the surface Σ. For example, if $\mathbf{f}$ represents the velocity field of a fluid, then the flux is the net quantity of fluid to flow through the surface Σ per unit time. A positive flux means there is a net flow *out* of the surface (i.e. in the direction of the outward unit normal vector $\mathbf{n}$), while a negative flux indicates a net flow inward (in the direction of $-\mathbf{n}$).

The term divergence comes from interpreting $\operatorname{div} \mathbf{f}$ as a measure of how much a vector field "diverges" from a point. This is best seen by using another definition of $\operatorname{div} \mathbf{f}$ which is equivalent[4] to the definition given by formula (4.32). Namely, for a point (x,y,z) in $\mathbb{R}^3$,

$$\operatorname{div} \mathbf{f}(x,y,z) = \lim_{V \to 0} \frac{1}{V} \iint\limits_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma} , \tag{4.33}$$

[3]See TAYLOR and MANN, § 15.6 for the details.

[4]See SCHEY, p. 36-39, for an intuitive discussion of this.

where V is the volume enclosed by a closed surface Σ around the point (x, y, z). In the limit, $V \to 0$ means that we take smaller and smaller closed surfaces around (x, y, z), which means that the volumes they enclose are going to zero. It can be shown that this limit is independent of the shapes of those surfaces. Notice that the limit being taken is of the ratio of the flux through a surface to the volume enclosed by that surface, which gives a rough measure of the flow "leaving" a point, as we mentioned. Vector fields which have zero divergence are often called *solenoidal* fields.

The following theorem is a simple consequence of formula (4.33).

Theorem 4.9. If the flux of a vector field $\mathbf{f}$ is zero through every closed surface containing a given point, then $\operatorname{div} \mathbf{f} = 0$ at that point.

Proof: By formula (4.33), at the given point (x, y, z) we have

$$\begin{aligned} \operatorname{div} \mathbf{f}(x,y,z) &= \lim_{V\to 0} \frac{1}{V} \iint_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma} \quad \text{for closed surfaces } \Sigma \text{ containing } (x,y,z), \text{ so} \\ &= \lim_{V\to 0} \frac{1}{V}(0) \quad \text{by our assumption that the flux through each } \Sigma \text{ is zero, so} \\ &= \lim_{V\to 0} 0 \\ &= 0\,. \end{aligned}$$

QED

Lastly, we note that sometimes the notation

$$\oiint_{\Sigma} f(x,y,z)\,d\sigma \quad \text{and} \quad \oiint_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma}$$

is used to denote surface integrals of scalar and vector fields, respectively, over closed surfaces. Especially in physics texts, it is common to see simply $\oint_{\Sigma}$ instead of $\oiint_{\Sigma}$.

Exercises

A

For Exercises 1-4, use the Divergence Theorem to evaluate the surface integral $\iint_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma}$ of the given vector field $\mathbf{f}(x, y, z)$ over the surface Σ.

1. $\mathbf{f}(x,y,z) = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$, $\Sigma : x^2 + y^2 + z^2 = 9$

2. $\mathbf{f}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, Σ : boundary of the solid cube $S = \{(x,y,z) : 0 \le x, y, z \le 1\}$

3. $\mathbf{f}(x,y,z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$, $\Sigma : x^2 + y^2 + z^2 = 1$

4. $\mathbf{f}(x,y,z) = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $\Sigma : x^2 + y^2 + z^2 = 1$

B

5. Show that the flux of any constant vector field through any closed surface is zero.

6. Evaluate the surface integral from Exercise 2 *without* using the Divergence Theorem, i.e. using only Definition 4.3, as in Example 4.10. Note that there will be a different outward unit normal vector to each of the six faces of the cube.

7. Evaluate the surface integral $\iint_\Sigma \mathbf{f} \cdot d\boldsymbol{\sigma}$, where $\mathbf{f}(x,y,z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ and Σ is the part of the plane $6x + 3y + 2z = 6$ with $x \ge 0$, $y \ge 0$, and $z \ge 0$, with the outward unit normal $\mathbf{n}$ pointing in the positive z direction.

8. Use a surface integral to show that the surface area of a sphere of radius r is $4\pi r^2$. (*Hint: Use spherical coordinates to parametrize the sphere.*)

9. Use a surface integral to show that the surface area of a right circular cone of radius R and height h is $\pi R\sqrt{h^2+R^2}$. (*Hint: Use the parametrization* $x = r\cos\theta$, $y = r\sin\theta$, $z = \frac{h}{R}r$, *for* $0 \le r \le R$ *and* $0 \le \theta \le 2\pi$.)

10. The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ can be parametrized using *ellipsoidal coordinates*

$$x = a\sin\phi\cos\theta \ , \ y = b\sin\phi\sin\theta \ , \ z = c\cos\phi \ , \quad \text{for } 0 \le \theta \le 2\pi \text{ and } 0 \le \phi \le \pi.$$

Show that the surface area S of the ellipsoid is

$$S = \int_0^\pi \int_0^{2\pi} \sin\phi\sqrt{a^2b^2\cos^2\phi + c^2(a^2\sin^2\theta + b^2\cos^2\theta)\sin^2\phi}\, d\theta\, d\phi \ .$$

(Note: The above double integral can not be evaluated by elementary means. For specific values of a, b and c it can be evaluated using numerical methods. An alternative is to express the surface area in terms of *elliptic integrals*.[5])

C

11. Use Definition 4.3 to prove that the surface area S over a region R in $\mathbb{R}^2$ of a surface $z = f(x,y)$ is given by the formula

$$S = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA \ .$$

(*Hint: Think of the parametrization of the surface.*)

[5] Bowman, F., *Introduction to Elliptic Functions, with Applications*, New York: Dover, 1961, § III.7.